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Determinate and random processes in cyclic and dynamic systems

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Abstract. A method for the description of fluctuations (variations) in the period of the motion of cyclic-dynamic systems elements is presented. The random change of the period of cyclic mechanisms, high-Q oscillators and auto-oscillating systems, as well as random changes of the time intervals between the moments of passing fixed linear co-ordinates by the moving mirror of a Fourier spectro-radiometer is studied. It is shown that the period of the fluctuations follows from a non-Markov random process.

Key words: fluctuations, generic function, motion, non-Markov random process, period, time

1. Introduction

The traditional description of dynamic systems consists in defining the change dependence of parameters characterising their state versus time. This way, with the help of the dynamic equations, we define the dependencies of the x-co-ordinate that describes the state of the system versus time t. We assume that the time run is uniform, and the time intervals Δt between the regular moments of determining the value of the x-co-ordinate are constant (Figure 1).

Still, in many problems associated with the description of periodic processes and motions of cyclic systems there is a necessity of determining time moments t corresponding to specific discrete values of the x-co-ordinate, *i.e.*, the $t(x)$ dependence (Figure 2).

In this case the time moments between the states of the system characterised by a discrete set of values of the x-co-ordinate are not equal and contain determinate and random components in the form of variations and fluctuations of the time intervals. The presence of such variations (or fluctuations) is to be considered when developing models of dynamic systems in $t(x)$ variables, and the ensuing peculiarities are to be studied adequately.

The above problem arises most prominently when studying the dynamics of the components of cyclic mechanisms [1, Chapter 3–6]. Variations (fluctuations) of the cycle of motions of their components appear due to the influence of determinate and random processes in such mechanisms. Actual measurements performed for mechanisms confirm the presence of the above nonuniformity of motion. Therefore, we believe it is justified to use the model while accounting for nonuniform changes of the time intervals when describing the dynamics of cyclic mechanisms. A similar problem also arises when describing controlling systems using time measurements of the motion parameters as the feedback.

A description of cyclic dynamic systems can be given in the simplest way when the relative fluctuations of the cycle of motion of their components are small. In this case it is possible to construct a linear theory linking fluctuations of the time intervals to the system state parameters. This in turn allows one to uniquely define statistic characteristics of the period fluctuations depending on the character of dynamic parameter fluctuations. Still, even for the

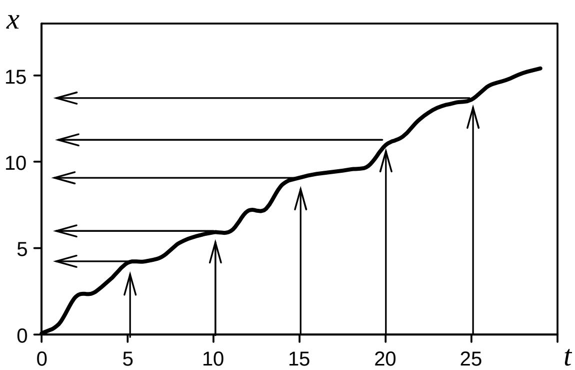

Figure 1. Dependence of the x-co-ordinate versus time t.

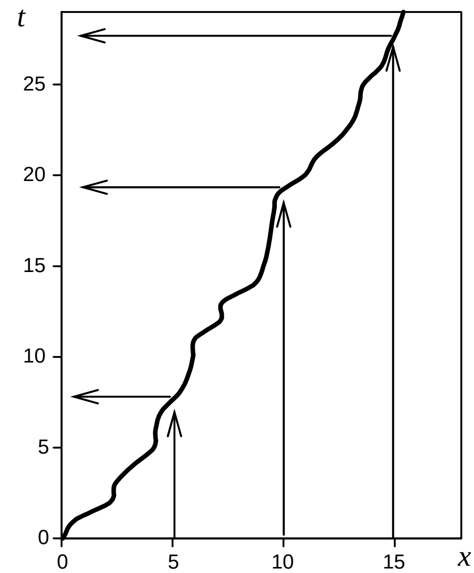

Figure 2. Dependence of time moments t versus x-co-ordinate.

above simplest case there arises a need to use non-Markov random processes to describe fluctuations of the oscillation period. This makes it more difficult to develop a theory.

2. Fluctuations of the revolution period of a cyclic dynamic-system shaft

Different types of reducers, electric engines, generators and other machines refer to cyclic dynamic systems with rotating components (shafts). Their specific feature is the presence of shafts that rotate nonuniformly, which reflects the specifics of the dynamic processes occurring in them.

In order to give a statistic description of the measurements of the time intervals characterising the passing of the shaft through a fixed angular position, it is necessary to construct dependencies that uniquely link the fluctuations (variations) of the angular displacement of the rotating component to the fluctuations (variations) of the time intervals of their rotation cycles. Constructing the above dependencies allows one to define time and spectral Hanning windows for the transition of the angular displacement into time intervals. In their turn these Hanning windows help to determine the principal limitations of using time measurements to study the dynamics of the cyclic performance of the mechanisms.

Let us define dependencies of the period fluctuations versus fluctuations of the angular displacement of the shaft. To determine the current period $T(t)$ of the rotation of the cyclic-mechanism component, it is possible to apply the following correlation [2, Equation 2]:

$$2\pi = \int_{t}^{t+T(t)} \dot{\varphi}(\tau)\mathrm{d}\tau, \tag{2.1}$$

where $\varphi(t)$ is the dependence of the shaft's angular displacement on time.

Considering that

$$T(t) = T_0 + \delta T(t), \tag{2.2}$$

where

$$|\delta T(t)| \ll T_0, \tag{2.3}$$

and $\delta T(t)$ defines fluctuations (variations) of the period, we may write expression (2.1) in a first approximation as:

$$2\pi = \omega_0 T(t) + \int_t^{t+T_0} \delta\dot{\varphi}(\tau)\mathrm{d}\tau. \tag{2.4}$$

Here we have

$$\varphi(t) = \omega_0 t + \delta\varphi(t), \tag{2.5}$$

where $\delta\varphi(t)$ defines fluctuations (variations) of the shaft's angular displacement and $\omega_0 = 2\pi/T_0$ is the average cyclic frequency. Here the assumption has been made that the mean square deviation of the fluctuation (variation) rate of the angle, $\sigma_{\delta\dot{\varphi}}$, is infinitesimal in comparison with the average cyclic frequency ω_0: $\sigma_{\delta\dot{\varphi}} \ll \omega_0$.

Fluctuations of the angular displacement and the period are due to noises and variations – due to weak periodic changes in the dynamics of the cyclic mechanism. Still, despite the variety of potential variations and fluctuations of the angular displacement and the period, it is possible to give their general mathematical description in the class of random functions. Therefore, in what follows the stochastic connectivity between the angular displacement and time intervals is analysed.

Equation (2.4) allows one to write down a linear correlation linking the period fluctuations (variations) $\delta T(t)$ and the angular fluctuations (variations) $\delta\varphi(t)$ as follows:

$$\delta T(t) = -\frac{T_0}{2\pi}(\delta\varphi(t+T_0) - \delta\varphi(t)). \tag{2.6}$$

Equation (2.6) allows one to understand the type of spectral Hanning window for transferring angular fluctuations into the current period fluctuations. If we assume that (2.6) describes an ideal system with one input and one output, then the time transformation window (pulse characteristic) can be presented in the following way:

$$h(t) = -\frac{T_0}{2\pi}(\delta(t+T_0) - \delta(t)), \tag{2.7}$$

where $\delta(t)$ in the delta-function.

The system correlation function of the procedure to change the current period becomes as follows:

$$R_h(\tau) = \int_{-\infty}^{\infty} h(t)h(t+\tau)\mathrm{d}t = \frac{T_0^2}{(2\pi)^2}(2\delta(\tau) - \delta(\tau - T_0) - \delta(\tau + T_0)). \tag{2.8}$$

In this case correlation functions of the period fluctuations $R_{\delta T}(\tau)$ and the angular fluctuations $R_{\delta\varphi}(\tau)$ will be restricted by the dependence

$$R_{\delta T}(\tau) = \int_{-\infty}^{\infty} R_h(t) R_{\delta\varphi}(t+\tau)\mathrm{d}t. \tag{2.9}$$

Applying a direct Fourier transform of (2.8), we get a spectral transformation window (frequency characteristic):

$$G_{\delta T}(\omega) = \frac{T_0^2}{\pi^2}\sin^2\left(\frac{\omega T_0}{2}\right). \tag{2.10}$$

It follows from (2.10), if the following conditions are met

$$\omega T_0 = 2\pi k, \tag{2.11}$$

where $k = \overline{1, n}$, that the fluctuation amplitude of the rotation period tends to zero. For this reason influences with frequencies close to $\omega = 2\pi k / T_0$ do not cause responses, and therefore do not change the rotation period. In particular, during measurements of the current period there is no possibility to register processes at frequencies that are integer-valued and divisible by the average rotation frequency. Because of this the accuracy of the measurement of the current period is not dependent on the error of the pitch of the registered (discrete) angular positions of the shaft, and is only determined by errors in the time-interval measurements.

Equation (2.10) shows that the intensity of the time-interval fluctuations for the same fluctuations of the angular displacement is determined by the co-factor T_0^2/π^2. In a first approximation the linear Equation (2.6) has a limited range of application. This has to do with the possibility of applying the inequality (2.3). In order to define the range of application of the linear relation (2.6), we have carried out a digital modelling of the shaft's angular displacement versus the current time by the formula:

$$\varphi(t_k) = \omega_0 t_k + \delta\varphi(t_k), \quad k = 1, 2 \ldots$$

Here $\delta\varphi(t_k)$ denotes white Gauss noise with a dispersion σ_φ^2. A calculation of the current period based on the results of digital modelling was carried out using $T_n = t_{k(n)} - t_{k(n-1)}$, where $t_{k(n)}$ is the current time for the shaft passing the nth fixed angular position defined such that $\varphi(t_k) = 2\pi n$, $n = 1, 2 \ldots$. With the values of the current period we find the assessed value for the dispersion of the current period fluctuations $\sigma_{\delta T}^2$ and calculate the intensity of the equivalent phase fluctuations by invoking $\sigma_{\hat{\varphi}} = \omega_0 \sigma_{\delta T}$.

An analysis of the digital-modelling results indicates that, if the relative level of the time interval fluctuation does not exceed 10%, a first approximation for the solution of the integral relation (2.1) satisfactorily describes the link between the angular displacement and the time-interval fluctuations in the steady-state mode of the cyclic mechanism. Growth of the intensity of the angular-displacement fluctuations leads to a change of the function describing the distribution of the period fluctuations, and to low-frequency filtration in the spectral domain (Figure 3).

Therefore, application of the linear transformation of fluctuations of the shaft's angular displacement into time-interval fluctuations is possible after studying the dynamics of the cyclic mechanisms with reasonably uniform rotation of the elements. It is necessary to study (2.1) directly when studying a mechanism's period of shaft rotation involving significant changes (*e.g.* an internal-combustion engine).

Let us review the following modelling problem to study the specifics of the transformation of the angular displacement within the period of shaft rotation. Suppose we have a stiff shaft that is influenced by moments of external forces and viscous friction. A linear differential equation describing the dynamics of the rotary motion of such a shaft is:

$$J\ddot{\varphi} + R\dot{\varphi} = M(t), \tag{2.12}$$

where J is the moment of inertia of the shaft, R is the friction factor and $M(t)$ is the external torque.

Assuming that

$$M(t) = M_0 + \delta M(t), \tag{2.13}$$

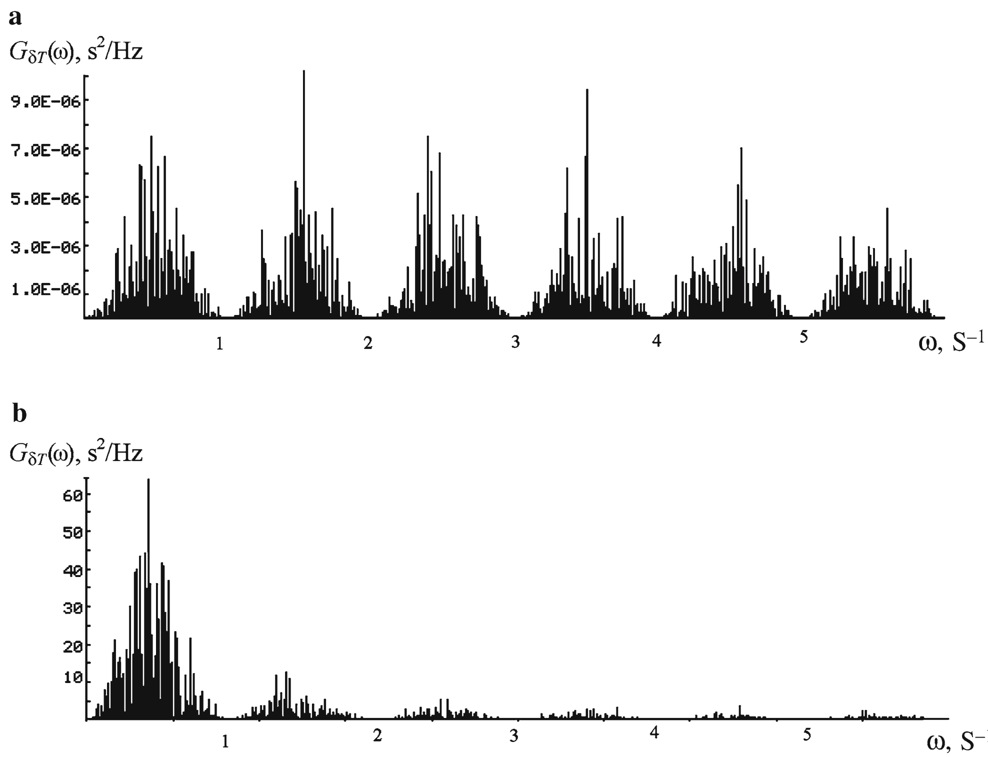

Figure 3. Spectral density of the power of the period fluctuations by 12 marks. $\omega_0 = 1\,\mathrm{s}^{-1}$; (**a**) $\sigma_\varphi = 10^{-3}$ rad, (**b**) $\sigma_\varphi = 10$ rad

where $|\delta M(t)| \ll M_0$, and $\delta M(t)$ denote the moment fluctuations, and inserting Equation (2.5) into Equation (2.12), we get the stochastic differential equation:

$$J\,(\delta\ddot{\varphi} + \alpha\delta\dot{\varphi}) = \delta M(t), \tag{2.14}$$

where $\alpha = R/J$ is a damping factor of the mechanical system. Multiplication of Equation (2.14) by $T_0/2\pi$ and considering (2.6) allow us to derive the following stochastic differential equation for the rotary motion of the shaft

$$J\left(\delta\ddot{T}(t) + \alpha\delta\dot{T}(t)\right) = -\frac{T_0}{2\pi}\left(\delta M(t+T_0) - \delta M(t)\right). \tag{2.15}$$

Suppose that fluctuations of the external torque $\delta M(t)$ are represented by white Gaussian noise (with zero distribution average), with a correlation function given by

$$R_{\delta M}(\tau) = \langle \delta M(t)\delta M(t+\tau)\rangle = D_m\delta(\tau), \tag{2.16}$$

where $\langle\ldots\rangle$ stands for the procedure to determine the distribution average and D_m is the bilateral spectral density of the power of external-torque fluctuations. Then the bilateral spectral density of the power of the angular displacement fluctuations becomes

$$G_{\delta\varphi}(\omega) = \frac{D_m}{J^2\left(\alpha^2\omega^2 + \omega^4\right)}, \tag{2.17}$$

and the bilateral spectral density of the power of the period fluctuations is equal to

$$G_{\delta T}(\omega) = \frac{D_m T_0^2}{\pi^2 J^2\left(\alpha^2\omega^2 + \omega^4\right)}\sin^2\left(\frac{\omega T_0}{2}\right). \tag{2.18}$$

To obtain Equation (2.18) we used the transformation spectral window (2.10). In the neighbourhood of zero frequency (2.18) indicates

$$G_{\delta T}(\omega)|_{\omega\to 0} = \frac{D_m T_0^4}{(2\pi)^2 \alpha^2 J^2}. \tag{2.19}$$

Figure 4 presents graphs of spectral densities of the power of phase fluctuations $G_{\delta\varphi}(\omega)$, and fluctuations of the current period $G_{\delta T}(\omega)$ of the shaft rotation expressed by (2.17) and (2.18) with the following parameters values: $J = 10^4\,\mathrm{kg\,m^2}$, $D_m = 10^8\,\mathrm{N^2\,m^2\,s}$, $\omega_0 = 314\,\mathrm{s^{-1}}$, $\alpha = 0{\cdot}4\,\mathrm{s^{-1}}$.

An analysis of the obtained dependencies of the spectral power densities $G_{\delta\varphi}(\omega)$ and $G_{\delta T}(\omega)$ indicates that the intensities of the angular-displacement fluctuations and of the period fluctuations increases along with the decrease of the shaft-rotation frequency. They reach their maximum values for zero frequency. The frequency $\omega = \alpha$ defines the limit of the low-frequency ($\omega < \alpha$) and high-frequency ($\omega > \alpha$) processes in the mechanical system. For the interval of frequencies of the angular-displacement fluctuations $[0, \alpha]$, the spectral density of the period-fluctuation power decreases as $G_{\delta T}(\omega) \sim \sin^2(\omega)/\omega^2$, and for the densities $[\alpha, \infty)$ as $G_{\delta T}(\omega) \sim \sin^2(\omega)/\omega^4$.

Therefore, the higher the frequency of the moment fluctuations at the shaft, the more the shaft limits them owing to its own moment of inertia J, thus reducing the intensity of the angular-displacement fluctuations and of the current rotation period.

The solution of differential equation (2.14) in the time domain yields the correlation functions of the frequency fluctuations $\delta\dot{\varphi}$, the angular displacement $\delta\varphi$, and the period of shaft revolution δT as:

$$R_{\delta\dot{\varphi}}(t_1, t_2) = \frac{D_m}{2\alpha J^2} \mathrm{e}^{-\alpha|t_2 - t_1|}, \tag{2.20}$$

$$R_{\delta\varphi}(t_1, t_2) = \frac{D_m}{2\alpha^3 J^2}\left[2\alpha \min(t_1, t_2) - \mathrm{e}^{-\alpha|t_2-t_1|} + \mathrm{e}^{-\alpha t_1} + \mathrm{e}^{-\alpha t_2} - 1\right], \tag{2.21}$$

$$\begin{aligned} R_{\delta T}(t_1, t_2) = \frac{D_m T_0^4}{2(2\pi)^2 \alpha J^2} & [2\alpha(2\min(t_1, t_2) - \min(t_1 + T_0, t_2) - \min(t_1, t_2 + T_0)) - \\ & - 2\mathrm{e}^{|t_2 - t_1|} + \mathrm{e}^{|t_2 - t_1 - T_0|} + \mathrm{e}^{|t_2 - t_1 + T_0|}\Big]. \end{aligned} \tag{2.22}$$

Equation (2.20) indicates that the rotation frequency is a steady-state random process, the correlation time of which is determined by the parameter α^{-1}. An analysis of (2.21)

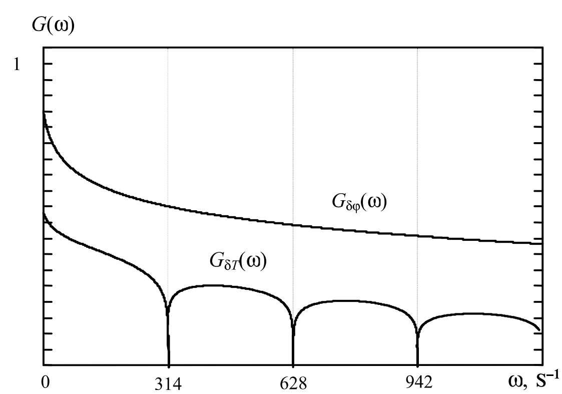

Figure 4. Spectral densities of the phase-fluctuation powers $G_{\delta\varphi}(\omega)$ rad^2/Hz and fluctuations of the shaft current rotation period $G_{\delta T}(\omega)$ s^2/Hz.

and (2.22) indicates that fluctuations of the angular displacement and of the period are non-Markov random processes, because their correlation functions depend not only on the difference of the selected time instants $t_2 - t_1$, but also on the history of the shaft's rotary motion all the way from start-up.

3. Fluctuations of the period of the dynamic-system oscillation

The fluctuations of the rotation period of the cyclic mechanism elements reviewed above can not be directly transferred to the study of dynamic systems with fluctuating elements. This is due to the fact that the rate of movement of the information label versus detector changes and is dependent on the displacement value of the oscillatory element with respect to the equilibrium position.

Therefore, when setting ourselves to the task of studying the dynamics of such mechanical systems as a torsion balance or a clockwork by measuring the oscillation period, we have to face the problem of constructing hardware functions for the transformation of fluctuations of the linear and angular displacements of oscillatory elements into fluctuations of their oscillation period.

Let us review the solutions of this problem. Suppose a variable $x(t)$ describes the linear displacement from the equilibrium position of an element in oscillatory motion. The condition that allows determining the current period of oscillations can be expressed as [1, Equation 4.1; 3, p. 11]:

$$x(t) = x(t + T(t)), \tag{3.1}$$

where $T(t)$ is the current oscillation period.

Further supposing that the oscillations are close to harmonic and their period can be represented as (2.2), (2.3), then let us, in a first approximation, present expression (3.1) as:

$$x(t) = x(t + T_0) + \dot{x}(t + T_0)\,\delta T(t). \tag{3.2}$$

Let us write the expression for the $x(t)$ co-ordinate as:

$$x(t) = x_0 \sin(\omega_0 t + \alpha_0) + \delta x(t), \tag{3.3}$$

where x_0 is the oscillation amplitude, α_0 the initial phase, $\delta x(t)$ fluctuations (variations) of the oscillatory element co-ordinate, and $\omega_0 = 2\pi/T_0$ the average cyclic oscillation frequency. Then, disregarding the infinitesimal higher-order terms, we may transform (3.2) into the correlation-binding fluctuations (variations) of the oscillation period $\delta T(t)$ and fluctuations (variations) of the co-ordinate $\delta x(t)$ as follows:

$$\delta T(t) = -\frac{T_0}{2\pi}\frac{1}{\sqrt{x_0^2 - x_n^2}}\left(\delta x(t + T_0) - \delta x(t)\right), \tag{3.4}$$

where $x_n = x(t)$ are the co-ordinates of the information detectors that register the occurrence of specified displacements of the oscillatory element at time t. It is important to note that the time moments t are not arbitrary here. They correspond to the moments when the oscillatory element passes the discrete positions of the information detectors x_n.

Comparison of (3.4) with expression (2.6) performed when studying cyclic mechanisms with elements that are in oscillatory motion has indicated that period fluctuations depend on the co-ordinates of the information detectors. It is also important to note that the measurement procedure that is based on defining the time intervals between the closest marks is quite complicated in the case of oscillatory motion, because it is then necessary to use a nonlinear transformation with the coefficient changing with time.

Similar transformations allow obtaining the correlation-binding fluctuations (variations) of the period $\delta T(t)$ and binding fluctuations (variations) of the angular displacement $\delta\varphi(t)$ for an element carrying out angular oscillations, *e.g.* for the weighing beam of a torsion balance or a clockwork, *viz.*

$$\delta T(t) = -\frac{T_0}{2\pi}\frac{1}{\sqrt{\varphi_0^2-\varphi_n^2}}\left(\delta\varphi(t+T_0)-\delta\varphi(t)\right), \tag{3.5}$$

where φ_0 is the amplitude of the element's angular oscillations, and $\varphi_n=\varphi(t)$ are the angular co-ordinates of the information detectors.

Expression (3.5) can be converted to (2.6) by multiplication of the measured fluctuations (variations) of the oscillation period by the scaling multiplier $\sqrt{\varphi_0^2-\varphi_n^2}$. Each measured value $\delta T(t)$ is to be scaled by taking the angular co-ordinates of the information detectors into account. Similarly, to transform (3.4) into (2.6) it is necessary to multiply the measurement results by the value $\sqrt{x_0^2-x_n^2}$.

It should be noted that without the above scaling, it will be very difficult to directly use the measured values of the fluctuations (variations) of the period $\delta T(t)$, which is defined by (3.5) (or (3.4)) because the value of $\delta T(t)$ depends on the angular (linear) co-ordinates of the measurement detectors. To analyse the measurement results in this case it is necessary to perform additional studies of the specifics of the transformation of the fluctuations (variations) of $\delta\varphi(t)$ (or $\delta x(t)$) co-ordinate into fluctuations (variations) of the oscillation period $\delta T(t)$.

An expression for the spectral window of the transformation of the linear displacement fluctuations into the current period fluctuations can be obtained by using the technique described in Section 2. In a first approximation for $|x_n|\ll x_0$ and accurate to within a constant factor, the obtained expression will coincide with (2.10), *viz.*

$$G_{\delta T}(\omega)=\frac{T_0^2}{\pi^2}\frac{1}{x_0^2-x_n^2}\sin^2\left(\frac{\omega T_0}{2}\right). \tag{3.6}$$

Replacing in (3.6) the linear co-ordinate x by the angular co-ordinate φ, we obtain an expression for the spectral window for the angular oscillations of an element of a cyclic mechanism, namely

$$G_{\delta T}(\omega)=\frac{T_0^2}{\pi^2}\frac{1}{\varphi_0^2-\varphi_n^2}\sin^2\left(\frac{\omega T_0}{2}\right). \tag{3.7}$$

Therefore, it is possible to use the results obtained in Section 2 for a rotary motion to study the specifics of the fluctuations of the oscillation period. In this case it is only necessary to introduce the respective scale factor.

The obtained expressions describing the transformation procedure for the fluctuations (variations) of the angular displacement of an oscillating element into fluctuations (variations) of the oscillation period can be used to analyse the capabilities of the dynamic-measurement method when applied in the case of gravity and seismic measurements with the help of a torsion balance [4, pp. 50–64, 86–88]. Measurements with the help of a torsion balance do not only take place in the static mode [5, pp. 356–387] when we register the twist angle of a thread, but also in the dynamic mode by registering the current period of oscillations [6, pp. 112–133; 7, pp. 88–89; 8, pp. 122–123]. In the latter case the information detectors are located at some fixed angles φ_n. They help to register the current oscillation period of the sensitive element of the torsion balance.

Let us describe the oscillations of the weighing beam of a torsion balance under the influence of external disturbances with the help of the equation

$$\ddot{\varphi} + 2\beta\dot{\varphi} + \omega_0^2\varphi = \xi(t), \tag{3.8}$$

where $\xi(t)$ represents white noise with dispersion D.

The procedure of dynamic measurements with a torsion balance assumes an initial excitation of the balance by a preliminary swinging through some φ_0 angle. For a high-quality balance carrying out natural oscillations, it is possible to assume that, within time intervals much shorter than the constant of fall-time oscillations of the weighing beam, the oscillations are close to harmonic. If the infinitesimal condition of fluctuations (variations) of the twist angle of the balance thread is met in comparison with the oscillation amplitude of the weighing beam φ_0, that is, $|\delta\varphi(t)| \ll \varphi_0$, it becomes possible to use the spectral window (3.7).

Considering that $\xi(t)$ is white noise, the spectral density of the angular fluctuations $\delta\varphi(t)$ can be presented as:

$$G_{\delta\varphi}(\omega) = \frac{D}{\left(\omega^2 - \omega_0^2\right)^2 + 4\beta^2\omega^2}, \tag{3.9}$$

and the respective spectral density of fluctuations of the oscillation period $\delta T(t)$ is given as:

$$G_{\delta T}(\omega) = \frac{D}{\left(\omega^2 - \omega_0^2\right)^2 + 4\beta^2\omega^2} \frac{T_0^2}{\pi^2} \frac{1}{\varphi_0^2 - \varphi_n^2} \sin^2\left(\frac{\omega T_0}{2}\right). \tag{3.10}$$

The expression (3.10) allows one to perform calculations of the spectral density of fluctuations of the oscillation period of the sensitive element of the torsion balance on condition that it is influenced by the external process described with the white-noise model. This expression can be used to assess the torsion-balance limiting sensitivity. During this type of measurement it is necessary to register oscillations of the balance weighing beam.

It should also be noted that, just as in case of studies of fluctuations of the rotation period of a cyclic-mechanism element for the analysis of the spectral density of fluctuations of the oscillation period of the sensitive element of the torsion balance, it is necessary to consider the number of zeros of the spectral density among frequencies that are integer-valued divisible by the average oscillation frequency. This specified feature leads in particular to the fact that, when we measure gravity and seismic impacts with the help of a torsion balance operating in the dynamic mode, they happen to be insensitive towards external disturbances of frequencies that are integer-valued divisible by the average oscillation frequency of the torsion pendulum. This limits the potential fields of application of the torsion balance with the preliminary excitation of oscillations of their sensitive element.

4. Dynamics of a control system that uses measurements of time intervals

It is necessary to develop adequate procedures to describe the dynamics of control systems that use the duration of the time interval between the moments of passing fixed displacements as the information parameter taken from the feedback detector. A system responsible for maintaining the rate of motion of the moving mirror of an infra-red measuring interferometer of a Fourier spectro-radiometer [9, p. 86; 10, p. 37; 11, pp. 32–34; 12, pp. 119–122] is an example of such a control system. Requirements imposed on the permissible instability of the mirror rate of motion are strict. Therefore, it is necessary to analyse the functioning of the reviewed control system with the specific features connected with the procedure of time measurements taken into account.

The method of measuring the rate of the mirror is based on registering the duration of its passing fixed linear displacements equal to half the length of the laser radiation wave λ_0. The laser is used in the reference channel of the Fourier spectro-radiometer. When the mirror moves a distance equal to $\lambda_0/2$, a shift of the fringe pattern by one interference band, a photoelectron detector develops an information signal. Registration of the time T between the moments when the detector is activated allows one to determine the rate of motion of the moving mirror $\dot{X} = \lambda_0/2T$. At a constant rate of the mirror movement V_0, the time interval between recurrent moments of passing by the moving mirror for fixed linear co-ordinates can be written as follows:

$$\lambda_0/2 = \int_{t-T(t)}^{t} \dot{X}(t)\,\mathrm{d}t. \tag{4.1}$$

The difference of the time intervals $T(t)$ of passing by the moving mirror of distances between recurrent activations of the detector and the value $T_0 \delta T(t) = T(t) - T_0$, can serve as the error signal in the feedback of the control system. In a first approximation, considering the rate of motion of the moving mirror close to the constant value V_0, the above difference can be presented as:

$$\delta T(t) = -\left(\left(X(t) - X(t - T_0)\right) - \lambda_0/2\right)/V_0, \tag{4.2}$$

where $X(t)$ denotes the dependence of the mirror co-ordinate as a function of time.

The motion of the mirror in the reviewed case can be described by the following equation [1, Equation 6.15]

$$\dddot{X} + 2\beta\ddot{X} + \omega^2\dot{X} + \gamma X = \kappa V_0 \delta T(t) + \xi_X(t), \tag{4.3}$$

where β, ω, γ and κ are parameters characterising the system and the control object; $\xi_X(t)$ stands for external determinate and random impacts. In Equation (4.3) the first component of the sum (summand) in the right-hand part describes the controlling impact of the control-system feedback.

Substitution of (4.2) in Equation (4.3) gives

$$\dddot{X}(t) + 2\beta\ddot{X}(t) + \omega^2\dot{X}(t) + (\gamma + \kappa) X(t) = \kappa X(t - T_0) + \kappa\lambda_0/2 + \xi_X(t). \tag{4.4}$$

Expression (4.4), taking into account (4.2), allows one to write down an equation describing variations (fluctuations) of the imbalance time $\delta T(t)$:

$$\delta\dddot{T}(t) + 2\beta\delta\ddot{T}(t) + \omega^2\delta\dot{T}(t) + (\gamma + \kappa)\delta T(t) = \kappa\delta T(t - T_0) + \gamma T_0 + \xi_T(t), \tag{4.5}$$

where

$$\xi_T(t) = -\left(\xi_X(t) - \xi_X(t - T_0)\right)/V_0. \tag{4.6}$$

Equation (4.5) shows that a stationary solution is obtained for $\delta T = T_0$ or $T = 2T_0$. This indicates that the principal peculiarity of the system that controls the motion of the mirror of the Fourier spectro-radiometer is the impossibility of ensuring a constant rate of mirror motion within an infinitely large section of its displacement. During the motion the δT values increase systematically, and the rate of motion of the moving mirror decreases. It should be noted that the obtained stationary solution does not describe the situation in general because Equation (4.5) is approximate and is correct only if $|\delta T(t)| \ll T_0$. Still the obtained result qualitatively coincides with experimental data [11, Figure 2], [12, Figure 3].

Based on the mathematical model (4.5) we performed a computational simulation of the moving mirror of a Fourier spectro-radiometer. The following parameters were used in the calculations: $T_0 = 25\,\mu$s, $V_0 = 1{\cdot}26 \times 10^{-2}$ m/s, $\beta = 30\,\text{s}^{-1}$, $\omega = 50\,\text{s}^{-1}$, $\gamma = 1{\cdot}5 \times 10^5\,\text{s}^{-3}$, $\kappa = 10^5\,\text{s}^{-2}$, and the mean square deviation of the random process $\xi_X(t)$ was assumed to be equal to $\sigma_{\xi_X} = 10^7\,\text{s}^{-3}$.

Figure 5 presents a calculated graph showing the dependence of the time interval T (between recurrent moments of the photoelectron detector actuation) on the movement of the mirror X. The initial transient that occurs after changing the direction of the mirror motion is clearly seen in the graph. The systematic trend connected with the growth of the time interval T, as long as the mirror moves, is also clearly seen in the graph. A comparison of this dependence with experimental data shows good agreement [1, Figure 6.1].

Figure 6 illustrates the spectrum $G_T(f)$ of fluctuations of the time interval $T(t)$ connected with the random influence onto the moving mirror. The presented graph shows the "hill" system typical for the time measurements that was thoroughly analysed in Section 2.

Therefore, the above description of the dynamics of the moving mirror, together with the control system that measures time intervals between recurrent moments of the photoelectron detector activation as the feedback allows one to calculate the specifics of how the specified technical system functions sufficiently adequately. Typical peculiarities that are revealed when measuring periods of cyclic motion of different mechanisms are also evident when studying the dynamics of the moving mirror of a Fourier spectro-radiometer.

5. Statistic description of the period fluctuations with the help of many-dimensional generic functions

The application of multi-dimensional generic functions is the most consistent method to give a statistic description of the periodic fluctuations for an arbitrary cyclic system. Multi-dimensional generic functions describing period fluctuations of a cyclic system allow one to obtain all the necessary statistic parameters of these fluctuations, such as n-dimensional distribution functions, correlation functions of the nth order, spectral densities, etc. In particular, n-dimensional generic functions allow the description of non-Markov processes in different physical environments [4, pp. 210–242], [13, pp. 265–267], [14, pp. 26–31], [15, pp. 1312–1314].

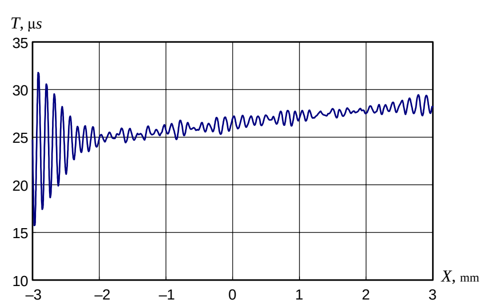

Figure 5. Dependence of the time interval T on the displacement of the moving mirror X.

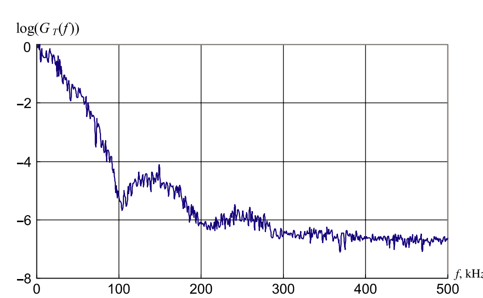

Figure 6. Spectral density $G_T(f)$ of the time-interval fluctuations $T(t)$.

Application of the method of multi-dimensional generic functions to describe period fluctuations of high-Q oscillators and self-oscillating systems statistically helps in finding the difference between the above fluctuations and fluctuations of the linear or angular displacement for these systems. Defining differences between the characteristics of period fluctuations and of the co-ordinate fluctuations that describe oscillations of the high-Q oscillator, is not only useful from an application's point of view, but is also of interest in fundamental research because the model of a high-Q oscillator is widely used for the description of different physical processes.

The methods reviewed above for finding statistic characteristics allow one to obtain correlation functions and spectral densities for fluctuations of the shaft rotation period and oscillations of the moving elements of cyclic mechanisms and measuring devices. Still, in order to solve some applied and fundamental problems, the necessity to have a more complete statistical description arises and, in particular, using multi-dimensional generic functions and multi-dimensional distribution functions [1, pp. 147–155], [16, pp. 80–85].

In accordance with [2, pp. 100–103] let us define multi-dimensional generic functions of the fluctuations $\delta T(t)$ of the rotation period based on *a priori* information about multi-dimensional generic functions of the angular-displacement fluctuations $\delta\varphi(t)$.

As was mentioned above, fluctuations of the rotation period of a cyclic-mechanism shaft $\delta T(t)$ are related to the angular displacement fluctuations $\delta\varphi(t)$ via Equation (2.6) as follows:

$$\delta T(t) = -\frac{T_0}{2\pi}(\delta\varphi(t+T_0) - \delta\varphi(t)), \tag{5.1}$$

where T_0 is the average value of the shaft rotation period. Equation (5.1) is correct when the period fluctuations $\delta T(t)$ are much less than the average period $T_0 : |\delta T(t)| \ll T_0$.

An analogous expression, accurate to within a constant factor $1/\varphi_0$, can also be obtained when interfacing fluctuations of the oscillation period $\delta T(t)$, and angular fluctuations $\delta\varphi(t)$, while measuring the period at the times when the oscillating system passes the equilibrium position (Equation (3.5)). In this case T_0 represents the average period of the cyclic-system oscillations.

For a cyclic system in fluctuating motion, for arbitrary placement of the measuring detectors, it is necessary to perform a scaling (as described in Section 2) by multiplying the measured period fluctuations by $\sqrt{\varphi_0^2 - \varphi_n^2}$ prior to processing the measurement results. This procedure allows one to obtain an interface between the angular displacement fluctuations $\delta\varphi(t)$ and the oscillation-period fluctuations $\delta T(t)$ as given by (5.1). A knowledge of the generic function $\delta\varphi(t)$ as given by (5.1) allows one to define the generic function of the process $\delta T(t)$.

Let us define the one-dimensional generic function $g(\lambda; t)$ describing the period fluctuations $\delta T(t)$. Putting the expression (5.1) into the general formula to define the generic function [13, p. 292, Formula 40] gives:

$$\begin{aligned} g(\lambda; t) &= \langle \exp(\mathrm{i}\lambda\delta T(t)) \rangle = \left\langle \exp\left(\mathrm{i}\frac{T_0}{2\pi}\lambda\delta\varphi(t) - \mathrm{i}\frac{T_0}{2\pi}\lambda\delta\varphi(t+T_0) \right) \right\rangle \\ &= \langle \exp(\mathrm{i}\mu_1\delta\varphi_1(\tau_1) + \mathrm{i}\mu_2\delta\varphi_2(\tau_2)) \rangle = h_2(\mu_1, \mu_2; \tau_1, \tau_2), \end{aligned} \tag{5.2}$$

where $\langle\cdots\rangle$ stands for the procedure to define the distribution average, λ is the Fourier image of the period fluctuations δT and μ is the Fourier image of the phase fluctuations $\delta\varphi$; further

$$\begin{aligned} &\mu_1 = \frac{T_0}{2\pi}\lambda, && \mu_2 = -\frac{T_0}{2\pi}\lambda, \\ &\delta\varphi_1(\tau_1) = \delta\varphi(t), && \delta\varphi_2(\tau_2) = \delta\varphi(t+T_0), \\ &\tau_1 = t, && \tau_2 = t + T_0. \end{aligned} \tag{5.3}$$

The function $h_2(\mu_1, \mu_2; \tau_1, \tau_2)$ is a two-dimensional generic function of a random process $\delta\varphi(t)$.

A similar argumentation allows one to define a n-dimensional generic function $g_n(\lambda_1, \dots, \lambda_n; t_1, \dots, t_n)$ of the process $\delta T(t)$ via the $2n$-dimensional generic function $h_{2n}(\mu_1, \dots, \mu_{2n}; \tau_1, \dots, \tau_{2n})$ of the process $\delta\varphi(t)$:

$$g_n(\lambda_1, \dots, \lambda_n; t_1, \dots, t_n) = h_{2n}(\mu_1, \dots, \mu_{2n}; \tau_1, \dots, \tau_{2n}), \tag{5.4}$$

where in the expression for the generic $h_{2n}(\mu_1, \dots, \mu_{2n}; \tau_1, \dots, \tau_{2n})$, it is necessary to have the following substitutions:

$$\begin{aligned} &\mu_j = \frac{T_0}{2\pi}\lambda_j, && \mu_{n+j} = -\frac{T_0}{2\pi}\lambda_j, \\ &\tau_j = t_j, && \tau_{n+j} = t_j + T_0, \qquad j = \overline{1, n}. \end{aligned} \tag{5.5}$$

Therefore, if the $2n$-dimensional generic function of the process $\delta\varphi(t)$ is known, then with the help of expression (5.4) it is possible to determine the n-dimensional generic function of the process $\delta T(t)$. In this case the order of magnitude of the generic function is reduced by a factor of two.

It should be noted that the expression (5.4) thus constructed does not allow backward transformation. This is because, in order to find the generic function of the process $\delta\varphi(t)$ with the help of the generic function of the process $\delta T(t)$, it is necessary to raise the order of magnitude of the generic function, which is impossible in general.

When using transformation (5.4) it is necessary to remember that for the succession of the time moments τ_j, $j = \overline{1, 2n}$ the requirement $\tau_j < \tau_{j+1}$ is optional. In actual fact, when the requirement $t_1 < t_2 < \dots < t_n$ is met, the conditions $\tau_1 < \tau_2 < \dots < \tau_n$ and $\tau_{n+1} < \tau_{n+2} < \dots < \tau_{2n}$ that follow from the expressions (5.5) are imposed on the succession of time moments τ_j. But, depending on the value of the average period T_0, the requirement $\tau_n < \tau_{n+1}$ is not indispensable and occurs only if $\tau_n - \tau_1 < T_0$.

If $\tau_n - \tau_1 > T_0$, the requirement $\tau_n < \tau_{n+1}$ is not met, even when applying Equation (5.4) when a $2n$-dimensional generic function $h_{2n}(\mu_1, \dots, \mu_{2n}; \tau_1, \dots, \tau_{2n})$ is imposed by the requirement $\tau_j < \tau_{j+1}$ and the necessity arises to rearrange the time moments τ_j with a respective correction of (5.5). The above correction is necessary when fluctuations of the value $\delta\varphi(t)$ are described in accordance with Wiener or Poisson random processes.

Let us now discuss some modelling examples regarding the definition of generic functions for the period fluctuations $\delta T(t)$ for different processes $\delta\varphi(t)$.

5.1. Example 1

Suppose a process $\delta\varphi(t)$ can be described as a Wiener random process with a four-dimensional generic function [13, pp. 176–179, Formula 34]:

$$h_4(\mu_1, \dots, \mu_4; \tau_1, \dots, \tau_4) = \exp\left[-\frac{1}{2}\nu \sum_{j,k=1}^{4} \mu_j \mu_k \min(\tau_j, \tau_k)\right], \tag{5.6}$$

where ν is the intensity of the process and $\min(\tau_j, \tau_k)$ is a procedure for determining the minimum value between τ_j and τ_k. Then the two-dimensional generic function of the process

$\delta T(t)$, when $t_2 > t_1$, can be presented as follows:

$$g_2(\lambda_1, \lambda_2; t_1, t_2) = \exp\left[-\frac{1}{2}\nu\frac{T_0^2}{(2\pi)^2}\left((\lambda_1+\lambda_2)^2 T_0 + 2\lambda_1\lambda_2 \min(t_2-t_1, T_0)\right)\right]. \tag{5.7}$$

Equation (5.7) is correct when $t_2 - t_1 < T_0$, and when $t_2 - t_1 > T_0$.

A transformation (see [4, Formula 7.49])

$$g_1(\lambda_1; t_1) = g_2(\lambda_1, \lambda_2; t_1, t_2)|_{\lambda_2=0} \tag{5.8}$$

allows one to write the one-dimensional generic function for the process under consideration as follows:

$$g_1(\lambda) = \exp\left[-\frac{1}{2}\nu\frac{T_0^3}{(2\pi)^2}\lambda^2\right]. \tag{5.9}$$

The correlation function of the period fluctuations $\delta T(t)$ can be defined with the help of the transformation (see [4, Formula 3.55])

$$R_{\delta T}(t_1, t_2) = \left.\frac{\partial^2 g_2(\lambda_1, \lambda_2; t_1, t_2)}{\mathrm{i}\partial\lambda_1 \mathrm{i}\partial\lambda_2}\right|_{\lambda_1=\lambda_2=0} \tag{5.10}$$

and in this case is as follows:

$$R_{\delta T}(t_2 - t_1) = \nu\frac{T_0^2}{(2\pi)^2}(T_0 - \min(t_2-t_1, T_0)), \tag{5.11}$$

where $t_2 > t_1$. Equation (5.11) indicates that, when $t_2 - t_1 \geq T_0$, the correlation function is $R_{\delta T}(t_2 - t_1) = 0$.

5.2. Example 2

Let us consider the case when $\delta\varphi(t)$ represents a general Poisson process with a four-dimensional generic function of the form (see [13, pp. 176–179, Formula 34]):

$$h_4(\mu_1, \ldots, \mu_4; \tau_1, \ldots, \tau_4) = \exp\left[\nu\sum_{j=1}^{4}\tau_j\left(g_a\left(\sum_{k=j}^{4}\mu_k\right) - g_a\left(\sum_{k=j+1}^{4}\mu_k\right)\right)\right], \tag{5.12}$$

where $g_a(\mu)$ is the generic function of the Poisson process discontinuity.

In this case the two-dimensional generic function of the process $\delta T(t)$ becomes:

$$\begin{aligned} g_2\left(\lambda_1, \lambda_2; t_1, t_2\right) = \exp\Bigg[-\nu\Bigg(\left(1 - g_a\left(-\frac{T_0}{2\pi}(\lambda_1+\lambda_2)\right)\right)T_0 + \\ + \left(1 - g_a\left(-\frac{T_0}{2\pi}\lambda_1\right) - g_a\left(-\frac{T_0}{2\pi}\lambda_2\right)\right. \\ \left. + g_a\left(-\frac{T_0}{2\pi}\left(\lambda_1+\lambda_2\right)\right)\right)\min\left(t_2 - t_1, T_0\right)\Bigg)\Bigg], \end{aligned} \tag{5.13}$$

where $t_2 > t_1$.

An application of transformation (5.8) yields the one-dimensional generic function

$$g_1(\lambda) = \exp\left[-\nu\left(\left(1 - g_a\left(-\frac{T_0}{2\pi}\lambda\right)\right)\ T_0\right)\right]. \tag{5.14}$$

If $\delta\varphi(t)$ represents a Poisson process with single discontinuities and are therefore, described by the four-dimensional generic function

$$h_4(\mu_1,\ldots,\mu_4;\tau_1,\ldots,\tau_4)=\exp\left[\nu\sum\nolimits_{j=1}^{4}\tau_j\left(\exp\left(i\sum\nolimits_{k=j}^{4}\mu_k\right)-\exp\left(i\sum\nolimits_{k=j+1}^{4}\mu_k\right)\right)\right], \tag{5.15}$$

then the two-dimensional generic function of the process $\delta T(t)$ is as follows

$$\begin{aligned} g_2(\lambda_1,\lambda_2;t_1,t_2)=\exp\Big[-\nu\Big(\Big(1-\exp\Big(-\mathrm{i}\frac{T_0}{2\pi}(\lambda_1+\lambda_2)\Big)\Big)T_0+ \\ +\Big(1-\exp\Big(-\mathrm{i}\frac{T_0}{2\pi}\lambda_1\Big)\Big)\Big(1-\exp\Big(-\mathrm{i}\frac{T_0}{2\pi}\lambda_2\Big)\Big)\min(t_2-t_1,T_0)\Big)\Big], \end{aligned} \tag{5.16}$$

and its one-dimensional generic function becomes

$$g_1(\lambda)=\exp\left[-\nu\left(\left(1-\exp\left(-\mathrm{i}\frac{T_0}{2\pi}\lambda\right)\right)T_0\right)\right]. \tag{5.17}$$

The correlation function for the reviewed event coincides with the function (5.11).

The two-dimensional generic functions of the process $\delta T(t)$ defined above allow one to conclude that, if transformation (5.1) is made to belong to a Wiener process, the ones for Poisson processes are transformed into processes that do not involve independent increments. For a process with an independent increment, the following equation is to be satisfied (see [13, p. 176, Formula 34]):

$$g_2(\lambda_1,\lambda_2;t_1,t_2)=\frac{g_1(\lambda_1+\lambda_2;t_1)\,g_1(\lambda_2;t_2)}{g_1(\lambda_2;t_1)}. \tag{5.18}$$

Substitution in Equation (5.18) of (5.7) and (5.9) does not lead to an identity. Similarly for expressions (5.13) and (5.14) Equation (5.18) is not an identity.

If the requirement $t_2-t_1\geq T_0$ is met, the considered processes become processes with independent values, because the following requirement is seen to be satisfied:

$$g_2(\lambda_1,\lambda_2;t_1,t_2)=g_1(\lambda_1;t_1)\,g_1(\lambda_2;t_2), \tag{5.19}$$

and the correlation function becomes zero (see Equation (5.11)).

5.3. Example 3

When the process $\delta\varphi(t)$ can be described by stationary Gauss noise with zero distribution average and the correlation function:

$$R_{\delta\varphi}(\tau_2-\tau_1)=D\exp(-\alpha|\tau_2-\tau_1|), \tag{5.20}$$

where D is the dispersion of the process $\delta\varphi(t)$ and α is the boundary frequency, then its four-dimensional generic function is:

$$h_4(\mu_1,\ldots,\mu_4;\tau_1,\ldots,\tau_4)=\exp\left[-\frac{D}{2}\sum\nolimits_{j,k=1}^{4}\mu_j\mu_k\exp\left(-\alpha\left|\tau_j-\tau_k\right|\right)\right]. \tag{5.21}$$

The two-dimensional generic function of the $\delta T(t)$ now becomes:

$$g_2(\lambda_1, \lambda_2; t_1, t_2) = \exp\left[-D\frac{T_0^2}{(2\pi)^2}\left\{ \left(\lambda_1^2+\lambda_2^2\right)(1-\exp(-\alpha T_0)) + \right.\right.$$
$$+\lambda_1\lambda_2 (2\exp(-\alpha\,|t_2-t_1|) - \exp(-\alpha\,|t_2-t_1-T_0|)$$
$$\left.\left. -\exp(-\alpha\,|t_2-t_1+T_0|))\right\}\right] \quad (5.22)$$

and its one-dimensional generic function is therefore equal to

$$g_1(\lambda) = \exp\left[-D\frac{T_0^2}{(2\pi)^2}\lambda^2(1-\exp(-\alpha T_0))\right]. \quad (5.23)$$

The correlation function of the $\delta T(t)$ period fluctuation in the event under consideration is:

$$R_{\delta T}(t_2-t_1) = D\frac{T_0^2}{(2\pi)^2}(2\exp(-\alpha|t_2-t_1|) - \exp(-\alpha|t_2-t_1-T_0|)$$
$$-\exp(-\alpha|t_2-t_1+T_0|)), \quad (5.24)$$

and the distribution average is equal to zero

If the process $\delta\varphi(t)$ represents white Gauss noise with the correlation

$$R_{\delta\varphi}(\tau_2-\tau_1) = D_{\delta\varphi}\delta(\tau_2-\tau_1), \quad (5.25)$$

where $D_{\delta\varphi}$ is the bilateral spectral density of the power of the fluctuations of the $\delta\varphi(t)$ process, the correlation function of the process $\delta T(t)$ becomes

$$R_{\delta T}(t) = D_{\delta\varphi}\frac{T_0^2}{(2\pi)^2}\left(2\delta(t) - \delta(t-T_0) - \delta(t+T_0)\right), \quad (5.26)$$

which corresponds to expression (2.8) obtained in Section 2.

Therefore, the examples considered here illustrate the change of the process characteristics when transferring from the angular fluctuations $\delta\varphi(t)$ to the period fluctuations $\delta T(t)$. Wiener and Poisson processes are transformed into processes that are not described by a process model with independent increments. Transformation (5.1) transforms a Markov random process of the phase fluctuations $\delta\varphi(t)$ described by expression (5.21) into a non-Markov random process of the period fluctuations $\delta T(t)$ with a two-dimensional generic function given by (5.22).

6. Fluctuations of the oscillation period of high-Q oscillator in case of resonant excitation

We shall now describe a high-Q oscillator that is influenced by a delta-correlated Gauss process. Nyquist noise can be viewed as such a process. We shall limit ourselves to the event when the high-Q oscillator is influenced not only by a random process but also by a resonant external force leading to the excitation of harmonic oscillations. Suppose that the amplitude of the oscillations caused by the influence of a random process is small in comparison with the amplitude of the determinate steady-state oscillations excited by an external resonant force.

For the situation consideration here an approximate description of the oscillations can be done with the help of an asymptotic method involving averaging within the oscillation period (see [17, pp. 230–237]).

The equation for the high-Q oscillations, when these are influenced by an external resonant force, can be presented as follows:

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = f(t) + \xi(t), \tag{6.1}$$

where β is a damping factor, ω_0 is the fundamental frequency of the oscillator's oscillations, $f(t)$ represents the external determinate harmonic influence with f_0 being the amplitude and ω_0 the frequency:

$$f(t) = f_0 \cos(\omega_0 t); \tag{6.2}$$

further $\xi(t)$ represents a random Gauss process with zero distribution average and correlation function as:

$$\langle \xi(t_2)\, \xi(t_1) \rangle = 2D\delta(t_2 - t_1). \tag{6.3}$$

The expression for the diffusion coefficient D depends on the type of oscillator (mechanical oscillating system, electric oscillatory circuit, etc.). For the case of a body with mass m suspended by an elastic chord, the expression is as follows:

$$D = \frac{2\beta k_B T_{tm}}{m}, \tag{6.4}$$

where k_B is Boltzmann's constant, T_{tm} is temperature.

Supposing that the requirement $\beta \ll \omega_0$ is satisfied for the oscillator, let us set up an asymptotic solution by applying the method of averaging (see [18, pp. 170–172]). To do this, we shall write the solution of Equation (6.1) as

$$x(t) = A(t) \sin(\omega_0 t + \delta\varphi(t)). \tag{6.5}$$

Then we obtain a system of abridged equations as [1, Equations 7.35, 7.36]:

$$\dot{A} + \beta A = \frac{f_0}{2\omega_0} \cos(\delta\varphi) + \xi_A(t), \tag{6.6}$$

$$A\delta\dot{\varphi} = -\frac{f_0}{2\omega_0} \sin(\delta\varphi) + \xi_{\delta\varphi}(t). \tag{6.7}$$

Here the functions $\xi_A(t)$ and $\xi_{\delta\varphi}(t)$ represent Gauss random processes with zero distribution average and correlation functions as [19, p. 77, Formula 2.63]:

$$\langle \xi_A(t_2)\, \xi_A(t_1) \rangle = \left\langle \xi_{\delta\varphi}(t_2)\, \xi_{\delta\varphi}(t_1) \right\rangle = \frac{D}{\omega_0^2} \delta(t_2 - t_1). \tag{6.8}$$

We now assume that the fluctuations of the oscillation phase $\delta\varphi(t)$ are small: $|\delta\varphi(t)| \ll 1$. Then, in a first approximation, the system of Equations (6.6) and (6.7) becomes

$$\dot{A} + \beta A = \frac{f_0}{2\omega_0} + \xi_A(t), \tag{6.9}$$

$$A\delta\dot{\varphi} = -\frac{f_0}{2\omega_0} \delta\varphi + \xi_{\delta\varphi}(t). \tag{6.10}$$

We write the amplitude of the oscillations $A(t)$ as the sum

$$A(t) = A_0 + \delta A(t), \tag{6.11}$$

where $\delta A(t)$ are small-amplitude fluctuations: $|\delta A(t)| \ll A_0$, $A_0 = \langle A(t) \rangle = \text{const}$ is the average value of the amplitude of the oscillations caused by the resonant excitation. Equation (6.9) yields the following:

$$A_0 = \frac{f_0}{2\beta\omega_0}. \tag{6.12}$$

Equation (6.10) becomes:

$$\delta\dot{\varphi} + \beta\delta\varphi = \frac{1}{A_0}\xi_{\delta\varphi}(t). \tag{6.13}$$

Equation (6.13) allows one to obtain a four-dimensional generic function of the process $\delta\varphi(t)$ (see [13, pp. 315–323]) which is similar to expression (5.21):

$$h_4(\mu_1, \ldots, \mu_4; \tau_1, \ldots, \tau_4) = \exp\left[-\frac{D}{4\beta\omega_0^2 A_0^2} \sum\nolimits_{j,k=1}^{4} \mu_j \mu_k \exp\left(-\beta\left|\tau_j - \tau_k\right|\right)\right]. \tag{6.14}$$

Suppose the period measurements occur at the time instants when the oscillator passes some positions x_n that correspond to the locations of the measuring detectors. In what follows we shall assume that the values of the period fluctuations $\delta T(t)$ are scaled through multiplication by the corresponding dimensionless coefficients $\sqrt{A_0^2 - x_n^2}/A_0$.

Then, in accordance with Equation (5.22), we obtain an expression for the two-dimensional generic function of the oscillation-period fluctuations $\delta T(t)$ of the high-Q oscillator with resonant excitation:

$$g_2(\lambda_1, \lambda_2; t_1, t_2) = \exp\left[-\frac{1}{2}\left(\left(\lambda_1^2 + \lambda_2^2\right) D_{\delta T} + 2\lambda_1\lambda_2 R_{\delta T}(t_2 - t_1)\right)\right], \tag{6.15}$$

where $D_{\delta T}$ is the dispersion of the period fluctuations:

$$D_{\delta T} = \frac{DT_0^4}{(2\pi)^4 \beta A_0^2}\left(1 - \exp(-\beta T_0)\right), \tag{6.16}$$

$R_{\delta T}(t_2 - t_1)$ is the correlation function:

$$\begin{aligned} R_{\delta T}(t_2 - t_1) = {} & \frac{DT_0^4}{2(2\pi)^4 \beta A_0^2}\left(2\exp\left(-\beta\left|t_2 - t_1\right|\right) - \right. \\ & \left. - \exp\left(-\beta\left|t_2 - t_1 - T_0\right|\right) - \exp\left(-\beta\left|t_2 - t_1 + T_0\right|\right)\right). \end{aligned} \tag{6.17}$$

Equation (6.15) allows one to obtain an expression for the two-dimensional distribution function of the oscillation period fluctuations $\delta T(t)$:

$$f_2(\delta T_1, \delta T_2; t_1, t_2) = \frac{1}{2\pi\sqrt{D_{\delta T}^2 - R_{\delta T}^2(t_2 - t_1)}} \exp\left[-\frac{\left(\delta T_1^2 + \delta T_2^2\right) D_{\delta T} - 2\delta T_1 \delta T_2 R_{\delta T}(t_2 - t_1)}{2\left(D_{\delta T}^2 - R_{\delta T}^2(t_2 - t_1)\right)}\right]. \tag{6.18}$$

It should be noted that expression (6.14) presented above allows us to write the equation for the two-dimensional distribution function of the oscillation-phase fluctuations $\delta\varphi(t)$ which is analogous to formula (6.18):

$$f_2(\delta\varphi_1, \delta\varphi_2; t_1, t_2) = \frac{1}{2\pi\sqrt{D_{\delta\varphi}^2 - R_{\delta\varphi}^2(t_2 - t_1)}} \exp\left[-\frac{\left(\delta\varphi_1^2 + \delta\varphi_2^2\right) D_{\delta\varphi} - 2\delta\varphi_1\delta\varphi_2 R_{\delta\varphi}(t_2 - t_1)}{2\left(D_{\delta\varphi}^2 - R_{\delta\varphi}^2(t_2 - t_1)\right)}\right], \tag{6.19}$$

where the phase-fluctuation dispersion is as follows:

$$D_{\delta\varphi} = \frac{DT_0^2}{2(2\pi)^2\beta A_0^2}; \tag{6.20}$$

the correlation function is therefore equal to

$$R_{\delta\varphi}(t_2 - t_1) = \frac{DT_0^2}{2(2\pi)^2\beta A_0^2} \exp(-\beta|t_2 - t_1|). \tag{6.21}$$

The correlation functions (6.17) and (6.21) allow one to obtain expressions for the bilateral spectral densities of the period fluctuations $\delta T(t)$ and the oscillation phase $\delta\varphi(t)$:

$$G_{\delta T}(\omega) = \frac{DT_0^4}{4(\pi)^4 A_0^2\left(\beta^2 + \omega^2\right)} \sin^2\left(\frac{\omega T_0}{2}\right), \tag{6.22}$$

$$G_{\delta\varphi}(\omega) = \frac{DT_0^2}{(2\pi)^2 A_0^2\left(\beta^2 + \omega^2\right)}. \tag{6.23}$$

Let us analyse the expressions obtained above. Figure 7 presents graphs of the spectral densities expressed by the Equations (6.22) and (6.23).

A comparative analysis of the presented graphs reveals a significant difference between fluctuations of the oscillation phase and the oscillation period. In the graphs of the spectral density $G_{\delta T}(\omega)$ describing oscillation-period fluctuations, frequencies with zero values of the spectral density are present. They are not present in the $G_{\delta\varphi}(\omega)$ graph. Analogous dependencies were obtained in Section 2 where we studied specific features of fluctuations of the rotation period of a cyclic-mechanism element.

7. Period fluctuations of a self-oscillating system

Let us now discuss an auto-oscillating system that is influenced by a delta-correlated Gauss random process. We assume that the system is in a mode of developed generation and that the amplitude of its oscillations is practically constant.

The equation describing the dynamics of the auto-oscillating system will be given as (see [17, p. 281, Equation 1.3], [20, Equation 7.2.9])

$$\ddot{x} + 2\beta\dot{x} + \kappa\dot{x}^3 + \omega_0^2 x = \xi(t), \tag{7.1}$$

where κ is a nonlinearity factor; also $\beta < 0$.

Let us construct abridged equations for the oscillation amplitude of the system and the oscillation phase:

$$\dot{A} + \beta A + \frac{3}{8}\kappa A^3 = \xi_A(t), \tag{7.2}$$

$$A\delta\dot{\varphi} = \xi_{\delta\varphi}(t). \tag{7.3}$$

As a first approximation the amplitude A_0 of the steady-state auto-oscillations can be found from Equation (7.2) as:

$$A_0 = \sqrt{\frac{8\,|\beta|}{3\kappa}}. \tag{7.4}$$

Then as an approximation the equation defining the phase fluctuations will be as follows:

$$\delta\dot{\varphi} = \frac{\xi_{\delta\varphi}(t)}{A_0}. \tag{7.5}$$

As in the previous paragraph we will assume that $\xi_{\delta\varphi}(t)$ is a differential coefficient of the Wiener process with the intensity

$$\nu = \frac{D}{\omega_0^2} \tag{7.6}$$

Then, in accordance with expression (5.7), the two-dimensional generic function of the period fluctuations $\delta T(t)$ will be the same (6.15), with

$$D_{\delta T} = \frac{D T_0^5}{(2\pi)^4 A_0^2}, \tag{7.7}$$

$$R_{\delta T}(t_2 - t_1) = \frac{D_{\delta T}}{T_0}\left(T_0 - \min\left(|t_2 - t_1|, T_0\right)\right). \tag{7.8}$$

The two-dimensional distribution function of the fluctuations of the auto-oscillation period will be described by expression (6.18) after substituting in it the dispersion value and the correlation function from (7.7) and (7.8).

If $|t_2 - t_1| < T_0$, this distribution function becomes:

$$f_2(\delta T_1, \delta T_2; t_1, t_2) = \frac{T_0}{2\pi D_{\delta T}\sqrt{(|t_2 - t_1|)(2T_0 - |t_2 - t_1|)}} \exp\left[-\frac{(\delta T_2 - \delta T_1)^2 T_0^2 + 2\delta T_1 \delta T_2 (|t_2 - t_1|) T_0}{2D_{\delta T}(|t_2 - t_1|)(2T_0 - |t_2 - t_1|)}\right]. \tag{7.9}$$

When $|t_2 - t_1| \geq T_0$, the two-dimensional distribution function is equal to a product of one-dimensional distribution functions:

$$f_2(\delta T_1, \delta T_2) = f_1(\delta T_1)\, f_1(\delta T_2) = \frac{1}{2\pi D_{\delta T}} \exp\left[-\frac{\delta T_1^2 + \delta T_2^2}{2D_{\delta T}}\right]. \tag{7.10}$$

The bilateral spectral density of the fluctuations of the auto-oscillation period is:

$$G_{\delta T}(\omega) = \frac{4D_{\delta T}}{\omega^2}\sin^2\left(\frac{\omega T_0}{2}\right). \tag{7.11}$$

The graph presented in Figure 8 illustrates the bilateral spectral density. It indicates that the oscillation-period fluctuations for an auto-oscillating system have a limited spectrum, even if the phase fluctuations represent a Wiener random process.

It should be noted that similar results are also obtained when analysing damped auto-oscillations of the linear oscillator within time intervals that are much shorter than the constant of the damping time: $t \ll 1/\beta$.

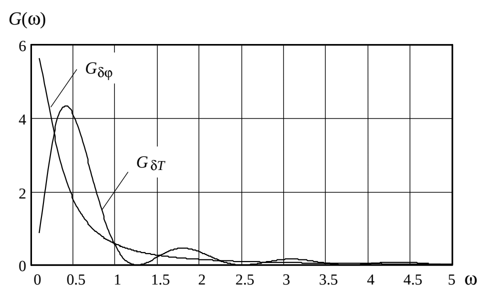

Figure 7. Graphs of spectral densities $G_{\delta T}(\omega)$ and $G_{\delta\varphi}(\omega)$.

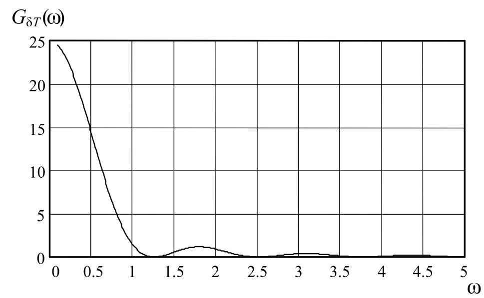

Figure 8. Graph of spectral density $G_{\delta T}(\omega)$.

If $\xi_{\delta\varphi}(t)$ is a differential coefficient of a Poisson random process with intensity ν that is defined by expression (7.6) then, in accordance with Equation (5.13), the two-dimensional generic function of the period fluctuations will become:

$$g_2(\lambda_1, \lambda_2; t_1, t_2) = \exp\left[-\frac{DT_0^3}{(2\pi)^2 A_0^2}\left(\left(1 - g_a\left(-\frac{T_0}{2\pi}(\lambda_1+\lambda_2)\right)\right)T_0 + \left(1 - g_a\left(-\frac{T_0}{2\pi}\lambda_1\right)\right.\right.\right.$$
$$\left.\left.\left. - g_a\left(-\frac{T_0}{2\pi}\lambda_2\right) + g_a\left(-\frac{T_0}{2\pi}(\lambda_1+\lambda_2)\right)\right)\min(|t_2 - t_1|, T_0)\right)\right]. \tag{7.12}$$

It may become necessary to apply (7.12) to calculate period fluctuations. This equation is needed to describe auto-oscillators, because noises that influence them are usually described by a Poisson random process.

8. Conclusions

The description of cyclic dynamic systems given here does not claim to be exhaustive. This is due to the fact that, when making models of systems with randomly changing time intervals, we only took into account events with insignificant fluctuations of these intervals. This assumption allowed us to obtain linear relations that could later become the basis for a linear theory of dynamic systems with fluctuating time. Moreover, the above theory was only applied for a sufficiently narrow category of technical devices.

Yet, the obtained results allowed us to solve the problem in a more general formulation. This could become feasible by performing an overall analysis and studies of the following specific problems.

In the first place it is necessary to construct methods for developing nonlinear relations between fluctuations of angular (or linear) displacements and fluctuations of the periods of cyclic systems. These relations allow the modelling of cyclic mechanisms for sufficiently big changes in the periods of their cyclic motion.

In the second place, the performed studies of time-interval fluctuations carried out here led to the problem of developing adequate methods of their statistic description while accounting for the non-Markov character of the above fluctuations. A model of the period fluctuations as a non-Markov random process should make it possible to perform a more adequate simulation of technical systems.

Thirdly, it is necessary to apply the developed approaches to other cyclic dynamic systems.

The proposed trends for further development of methods to describe cyclic dynamic systems does not cover all potential problems. Even so, their solution can give additional information about processes taking place in technical devices, and will ensure the development of adequate methods to study and describe complex dynamic systems.

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